

Bogoliubov Coefficients of 2D Charged Black Holes

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Abstract

We exactly calculate the thermal distribution and temperature of Hawking radiation for a two-dimensional charged dilatonic black hole after it has settled down to an “equilibrium” state. The calculation is carried out using the Bogoliubov coefficients. The background of the process is furnished by a preexisting black hole and not by collapsing matter as considered by Giddings and Nelson for the case of a Schwarzschild black hole. Furthermore, the vanishing of the temperature and/or the Hawking radiation in the extremal case is obtained as a regular limit of the general case.

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Introduction

By 1973, it was well known that black holes had two main properties: (i) the total surface of the event horizon cannot decrease and (ii) the surface gravity is constant on the event horizon. These properties were elements of the classical theory of black holes [1]. In 1974, S.W. Hawking observed that black holes radiate as a consequence of quantum effects and that the spectrum of this radiation is thermal [2]. Therefore, black holes can evaporate and in order to understand this process we are necessarily led to a “marriage” between quantum mechanics and gravity even in the presence of a weak gravitational field. The divergent basic assumptions underlying these two theories gave birth to a number of serious problems which hadn’t been faced till then. A characteristic paradox which has not yet been resolved is the “loss of information”. For all such issues, two-dimensional black holes can be a theoretical lab since they describe the spherically symmetric sector of the corresponding four-dimensional geometries. Moreover, in 1992 two-dimensional black hole solutions were seen as background solutions of effective actions emerging from string theory [3,4], leading to a more careful study of this field [5,6].

In this work, motivated by the calculation of the Bogoliubov coefficients in the background of a two-dimensional Schwarzschild black hole formed by the collapse of conformal matter [6], we calculate the corresponding effect in the background of a primordial charged black hole. The paper is organized as follows. In section 1 we repeat the calculation of Giddings and Nelson in the case of a primordial black hole. That is we do not invoke the region of the linear dilaton vacuum and we choose conveniently the ingoing states. The coincidence with the results of [6] proves the correctness of our choice. In section 2 we extend the analysis of section 1 to the case of a two-dimensional charged black hole. In particular we calculate the Bogoliubov coefficients, the resulting thermal distribution and the corresponding temperature. Our results are, of course, in accordance with what is known about charged black holes. One of the interesting outcomes of our calculation is that it yields regular continuous limit both for the case of an uncharged and an extremal black hole. Finally we conclude with a discussion of our results.

1 “Schwarzschild” Black Hole

The Bogoliubov coefficients for the case of a “Schwarzschild” two-dimensional black hole have been explicitly determined in [6]. The background of this process is taken to be collapsing matter. The formed black hole is characterized by a thermal distribution of Hawking radiation at temperature $T = \frac{\lambda}{2\pi}$. In this section we are going to compute

the Hawking radiation emitted by a “Schwarzschild” black hole without invoking the formation process. Before proceeding with the calculation of the Bogoliubov coefficients we will give the line element of the black hole in various coordinate systems. This is important since various parts of the calculations that follow are simplified by the use of different coordinate systems.

i) “Schwarzschild” gauge

The two-dimensional dilatonic black hole in the “Schwarzschild” gauge is characterized by the line element:

$$ds^2 = -g(r)dt^2 + g^{-1}(r)dr^2 \quad (1)$$

where the function $g(r)$ is given by:

$$g(r) = 1 - \frac{M}{\lambda} e^{-2\lambda r} \quad (2)$$

and $0 < t < +\infty$, $r_H < r < +\infty$, with $r_H = \frac{1}{2\lambda} \ln(\frac{M}{\lambda})$ the position of the event horizon of the black hole.

ii) Unitary gauge

The line element is:

$$ds^2 = -\tanh^2(\lambda y)dt^2 + dy^2 \quad (3)$$

where the “unitary” variable y is given by the following expression:

$$y = \frac{1}{\lambda} \ln[e^{\lambda(r-r_H)} + \sqrt{e^{2\lambda(r-r_H)} - 1}] \quad (4)$$

and $0 < y < +\infty$.

iii) Conformal gauge

The line element in this gauge is:

$$ds^2 = (1 + e^{-2\lambda x})^{-1}(-dt^2 + dx^2) \quad (5)$$

where the variable x is given by:

$$x = \frac{1}{2\lambda} \ln[e^{2\lambda(r-r_H)} - 1] \quad (6)$$

and $-\infty < x < +\infty$.

iv) Asymmetric gauge

The corresponding line element is:

$$ds^2 = -\frac{X}{X+1}dt^2 + \frac{dX^2}{4\lambda^2 X(X+1)} \quad (7)$$

where the “asymmetric” variable is given by:

$$X = e^{2\lambda(r-r_H)} - 1 \quad (8)$$

and $0 < X < +\infty$.

Considering now a massless scalar field Φ satisfying the Klein-Gordon equation $\square\Phi = 0$ we have for example in the unitary gauge:

$$-\frac{1}{\tanh^2(\lambda y)} \frac{\partial^2}{\partial t^2} \Phi(t, y) + \frac{1}{\tanh(\lambda y)} \frac{\partial}{\partial y} [\tanh(\lambda y) \frac{\partial \Phi(t, y)}{\partial y}] = 0 \quad (9)$$

The solution for the modes of the scalar field (following E.C. Titchmarsh [7]) is given by:

$$\Phi_\omega(t, y) = N(\omega) e^{-i\omega t} [\sinh(\lambda y)]^{i\omega/\lambda}, \quad (10)$$

where the normalization factor reads:

$$N(\omega) = \frac{2^{i\frac{\omega}{2\lambda}}}{\sqrt{4\pi\omega}} \quad (11)$$

As it is well known [8] the Bogoliubov coefficients are given by:

$$\alpha_{\omega\omega'} = (u_\omega^{out}, u_{\omega'}^{in}) = -i \int_\Sigma u_\omega^{out}(t, y) \overset{\leftrightarrow}{\partial}_\mu u_{\omega'}^{in*}(t, r) \eta^\mu d\Sigma \quad (12)$$

$$\beta_{\omega\omega'} = -(u_\omega^{out}, u_{\omega'}^{in*}) = i \int_\Sigma u_\omega^{out}(t, y) \overset{\leftrightarrow}{\partial}_\mu u_{\omega'}^{in}(t, r) \eta^\mu d\Sigma \quad (13)$$

where u_ω^{out} , $(u_{\omega'}^{in})$ two complete sets of outgoing (incoming) states and the integration performed on a hypersurface. For our calculation we take as outgoing states the solutions in (10) while for incoming states we choose plane waves

$$u_\omega^{in}(t, r) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega t} e^{i\omega r}. \quad (14)$$

Working in the “Schwarzschild” gauge where $d\Sigma = dr$, $\eta^\mu = (-1, 0)$ the future directed timelike vector and identifying the variable r in (14) with the one of the corresponding gauge we can evaluate the integrals in (12, 13). A lengthy but straightforward calculation [9] yields for the Bogoliubov coefficients:

$$\alpha_{\omega\omega'} = -\frac{1}{4\pi\lambda} \sqrt{\frac{\omega'}{\omega - i\epsilon}} 2^{i\frac{\omega}{2\lambda}} \left(\frac{M}{\lambda}\right)^{-i\frac{\omega'}{2\lambda}} B\left(-i\frac{\omega}{2\lambda} + i\frac{\omega'}{2\lambda} + \epsilon, 1 + i\frac{\omega}{2\lambda}\right) \quad (15)$$

$$\beta_{\omega\omega'} = \frac{1}{4\pi\lambda} \sqrt{\frac{\omega'}{\omega - i\epsilon}} 2^{i\frac{\omega}{2\lambda}} \left(\frac{M}{\lambda}\right)^{-i\frac{\omega'}{2\lambda}} B\left(-i\frac{\omega}{2\lambda} - i\frac{\omega'}{2\lambda} + \epsilon, 1 + i\frac{\omega}{2\lambda}\right) \quad (16)$$

where $\epsilon > 0$ a positive quantity which is necessary in order that the expressions in (15, 16) be well defined [6], and B stands for the beta function. The corresponding probabilities, giving the spectrum of the black hole radiation are taken by squaring (15, 16) leading to:

$$|\alpha_{\omega\omega'}|^2 = \left(\frac{1}{4\pi\lambda}\right)^2 \left|\frac{\omega'}{\omega - i\epsilon}\right| \left|B\left(-i\frac{\omega}{2\lambda} + i\frac{\omega'}{2\lambda} + \epsilon, 1 + i\frac{\omega}{2\lambda}\right)\right|^2 \quad (17)$$

$$|\beta_{\omega\omega'}|^2 = \left(\frac{1}{4\pi\lambda}\right)^2 \left|\frac{\omega'}{\omega - i\epsilon}\right| \left|B\left(-i\frac{\omega}{2\lambda} - i\frac{\omega'}{2\lambda} + \epsilon, 1 + i\frac{\omega}{2\lambda}\right)\right|^2 \quad (18)$$

The above spectrum leads to a thermal distribution at temperature [8] :

$$T_H = \frac{\lambda}{2\pi}. \quad (19)$$

Note that the above result can be reached in any coordinate system just by taking care of the appropriate Jacobian. The coincidence with the results of [6] points to the fact that the careful use of the plane waves as incoming states is correct even if we have not the linear dilaton vacuum region which exists in [6]. We can thus proceed with the consideration of the charged black hole.

2 Charged Black Hole

The line element of the charged two-dimensional black hole is given by (in coordinates corresponding to the ‘‘Schwarzschild’’ gauge of the previous section) [10]:

$$ds^2 = -g(r)dt^2 + g^{-1}(r)dr^2 \quad (20)$$

where

$$g(r) = 1 - \frac{M}{\lambda}e^{-2\lambda r} + \frac{Q^2}{4\lambda^2}e^{-4\lambda r} \quad (21)$$

with $0 < t < +\infty$, $r_+ < r < +\infty$, r_+ being the future event horizon of the black hole.

Following a parametrization analogous to the four-dimensional case the metric function factorizes as:

$$g(r) = (1 - \rho_- e^{-2\lambda r})(1 - \rho_+ e^{-2\lambda r}) \quad (22)$$

where

$$\rho_{\pm} = \frac{M}{2\lambda} \pm \frac{1}{2\lambda} \sqrt{M^2 - Q^2} \quad (23)$$

we can recognize immediately the outer event horizon H^+ placed at the point $r_+ = \frac{1}{2\lambda} \ln \rho_+$, while the ‘‘inner’’ horizon H^- is at the point $r_- = \frac{1}{2\lambda} \ln \rho_-$. In the extremal case ($Q = M$) the two surfaces coincide in a single event horizon at the point:

$$r_H = \frac{1}{2\lambda} \ln \left(\frac{M}{2\lambda} \right). \quad (24)$$

The line element (20-23) takes the following forms in the different coordinate systems previously considered:

i) Unitary gauge

The line element reads

$$ds^2 = -\frac{\mu^2 \sinh^2(\lambda y) \cosh^2(\lambda y)}{[\mu \sinh^2(\lambda y) + 1]^2} dt^2 + dy^2 \quad (25)$$

where the unitary variable is given by

$$y = \frac{1}{\lambda} \ln \left[\sqrt{\frac{1}{\mu} (e^{2\lambda(r-r_+)} - 1)} + \sqrt{\frac{1}{\mu} (e^{2\lambda(r-r_+)} - 1) + 1} \right] \quad (26)$$

and $0 < y < +\infty$, while $\mu = 1 - \frac{\rho_-}{\rho_+}$.

ii) Asymmetric gauge

In this coordinate system the line element reads:

$$ds^2 = -\frac{X(X+\mu)}{(X+1)^2} dt^2 + \frac{dX^2}{4\lambda^2 X(X+\mu)} \quad (27)$$

where $X = e^{2\lambda(r-r_+)} - 1 = \frac{e^{2\lambda r}}{\rho_+} - 1$, and $0 < X < +\infty$.

iii) Conformal Gauge

The line element of the two-dimensional spacetime is:

$$ds^2 = \frac{X(x)[X(x)+\mu]}{[X(x)+1]^2} (-dt^2 + dx^2) \quad (28)$$

where the “conformal” variable x is given by:

$$x = \frac{1}{2\lambda} \ln \left[X^{\frac{1}{\mu}} (X+\mu)^{1-\frac{1}{\mu}} \right] \quad (29)$$

and $-\infty < x < +\infty$.

Considering again a massless scalar field Φ satisfying the Klein-Gordon equation $\square\Phi = 0$ in the conformal gauge, which in this case is more convenient for simplicity reasons, we get the solution for the modes of the scalar field :

$$\Phi_\omega(t, x) = N(\omega) e^{-i\omega t} e^{i\omega x} \quad (30)$$

where the normalization factor reads :

$$N(\omega) = \frac{1}{\sqrt{(2\pi)(2\omega)}} \quad (31)$$

Now we can follow the analysis of previous section. Our incoming states are again plane waves, with the space variable identified to the variable of the corresponding

”Schwarzschild” gauge. In this case it turns out that the asymmetric gauge is more convenient for evaluating the integrals in (12), (13); one needs only to use the appropriate Jacobian. Skipping the details of the calculation, the Bogoliubov coefficients for the case of two-dimensional charged black hole are found to be :

$$\alpha_{\omega\omega'} = -\frac{1}{4\pi\lambda} \sqrt{\frac{\omega'}{\omega - i\epsilon}} \rho_+^{-i\frac{\omega'}{2\lambda}} B \left[-i\frac{\omega}{2\lambda} + i\frac{\omega'}{2\lambda} + \epsilon, 1 + i\frac{\omega}{2\lambda\mu} \right] \times {}_2F_1 \left[-i\frac{\omega}{2\lambda} + i\frac{\omega}{2\lambda\mu}, -i\frac{\omega}{2\lambda} + i\frac{\omega'}{2\lambda}; -i\frac{\omega}{2\lambda} + i\frac{\omega'}{2\lambda} + i\frac{\omega}{2\lambda\mu} + 1; 1 - \mu \right] \quad (32)$$

$$\beta_{\omega\omega'} = \frac{1}{4\pi\lambda} \sqrt{\frac{\omega'}{\omega - i\epsilon}} \rho_+^{i\frac{\omega'}{2\lambda}} B \left[-i\frac{\omega}{2\lambda} - i\frac{\omega'}{2\lambda} + \epsilon, 1 + i\frac{\omega}{2\lambda\mu} \right] \times {}_2F_1 \left[-i\frac{\omega}{2\lambda} + i\frac{\omega}{2\lambda\mu}, -i\frac{\omega}{2\lambda} - i\frac{\omega'}{2\lambda}; -i\frac{\omega}{2\lambda} - i\frac{\omega'}{2\lambda} + i\frac{\omega}{2\lambda\mu} + 1; 1 - \mu \right] \quad (33)$$

Squaring we get the probabilities for particle emission:

$$|\alpha_{\omega\omega'}|^2 = \left(\frac{1}{4\pi\lambda} \right)^2 \left| \frac{\omega'}{\omega - i\epsilon} \right| \left| B \left[-i\frac{\omega}{2\lambda} + i\frac{\omega'}{2\lambda} + \epsilon, 1 + i\frac{\omega}{2\lambda\mu} \right] \right|^2 \times \left| {}_2F_1 \left[-i\frac{\omega}{2\lambda} + i\frac{\omega}{2\lambda\mu}, -i\frac{\omega}{2\lambda} + i\frac{\omega'}{2\lambda}; -i\frac{\omega}{2\lambda} + i\frac{\omega'}{2\lambda} + i\frac{\omega}{2\lambda\mu} + 1; 1 - \mu \right] \right|^2 \quad (34)$$

$$|\beta_{\omega\omega'}|^2 = \left(\frac{1}{4\pi\lambda} \right)^2 \left| \frac{\omega'}{\omega - i\epsilon} \right| \left| B \left[-i\frac{\omega}{2\lambda} - i\frac{\omega'}{2\lambda} + \epsilon, 1 + i\frac{\omega}{2\lambda\mu} \right] \right|^2 \times \left| {}_2F_1 \left[-i\frac{\omega}{2\lambda} + i\frac{\omega}{2\lambda\mu}, -i\frac{\omega}{2\lambda} - i\frac{\omega'}{2\lambda}; -i\frac{\omega}{2\lambda} - i\frac{\omega'}{2\lambda} + i\frac{\omega}{2\lambda\mu} + 1; 1 - \mu \right] \right|^2 \quad (35)$$

which are useful for the evaluation of the Hawking radiation for a two-dimensional charged black hole.

It is noteworthy to observe that if we set $\mu = 1$, (in other words $\rho_- = 0$, i.e. the black hole has no charge ($Q = 0$), then the expressions (32) and (33) coincide with expressions (15) and (16) respectively. It is also very interesting to observe the consequences of the presence of the electric charge to the “equilibrium” state in which the black hole has settled down and more specifically the relation between the electric charge and the temperature. If we wish to count the particles which are detected in the case of “Schwarzschild” black hole and charged black hole, we have to evaluate the following quantity :

$${}_{in}\langle 0 | N_{\omega}^{out} | 0 \rangle_{in} = \int_0^{+\infty} d\omega' |\beta_{\omega\omega'}|^2 \quad (36)$$

For our convenience and without loss of generality, we set $\lambda = \frac{1}{2}$ and the equation (35)

becomes :

$$|\beta_{\omega\omega'}|^2 = \left(\frac{1}{2\pi}\right)^2 \left|\frac{\omega'}{\omega - i\epsilon}\right|^2 \left|B\left[-i\omega - i\omega' + \epsilon, 1 + i\frac{\omega}{\mu}\right]\right|^2 \times \\ \left|{}_2F_1\left[-i\omega + i\frac{\omega}{\mu}, -i\omega - i\omega'; -i\omega - i\omega' + i\frac{\omega}{\mu} + 1; 1 - \mu\right]\right|^2 \quad (37)$$

Plotting the quantity (36) as a function of frequency ω of the outgoing modes for different values of the parameter μ (which denotes the presence of the electric charge and the ratio between mass M and charge Q), we get figure 1.

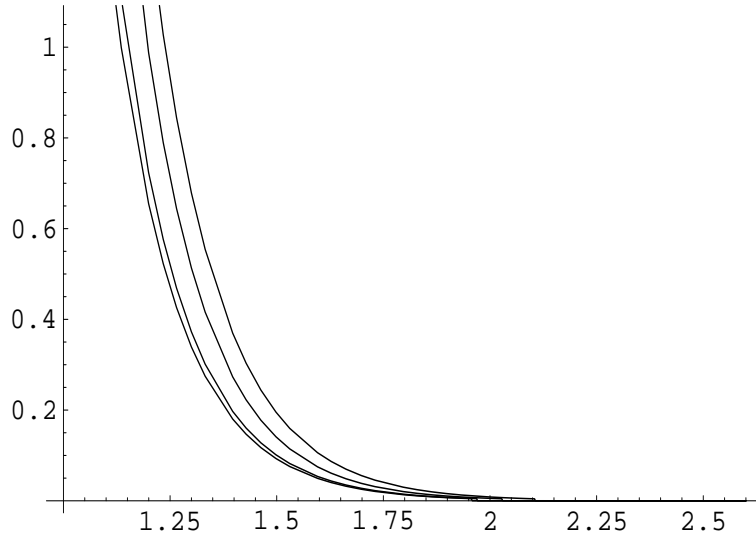


Figure 1: Diagrams obtained from the integration of the quantity (36) over the variable ω' for $\mu = 1$, $\mu = 0.9$, $\mu = 0.7$ and $\mu = 0.5$ respectively.

It is obvious from the above graph that as μ decreases, i.e. the ratio $\frac{Q}{M}$ increases, the graph “falls down”. This means that the corresponding black holes have lower temperature. This result can also be reached considering the definition [8]:

$$T_H = \frac{\kappa}{2\pi} \quad \text{and} \quad \kappa = \frac{1}{2} \left. \frac{\partial g(r)}{\partial r} \right|_{r=r_H} \quad (38)$$

which leads to the following expression for the Hawking temperature :

$$T_H = \frac{\lambda}{2\pi} \mu \quad (39)$$

Equation (39) is the “Reissner-Nordstrom” analog of equation (19) holding for the “Schwarzschild” case. When the parameter μ approaches 1, i.e. the electric charge is zero ($Q = 0$),

we get that (39) goes over to (19). When the parameter μ approaches zero, i.e. the electric charge is equal to mass ($Q = M$, extremal case), then the temperature is zero ($T = 0$), i.e. the black hole “freezes” completely. This known result [11–13] appears here as the regular ($\mu \rightarrow 0$ or $\mu \rightarrow 1$) limit of the more general case. This can be seen either from figure 1, or analytically working the limiting cases of (34), (35). The zero charge case has already been discussed. Furthermore the coefficients in the abovementioned expressions go to those of the flat space as $\mu \rightarrow 0$ indicating the “freezing” of the extremal black hole. Namely $|\alpha_{\omega\omega'}|^2 \rightarrow \delta_{\omega\omega'}$ and $|\beta_{\omega\omega'}|^2 \rightarrow 0$, for positive to positive frequency transitions.

3 Discussion

In this work, we have explicitly calculated the Bogoliubov coefficients for a background which is an already formatted (primordial) two-dimensional black hole with : (a) mass M (“Schwarzschild” case), (b) mass M and electric charge Q (“Reissner-Nordstrom” case). We have evaluated graphically their corresponding thermal spectrums and we have found an explicit expression for their temperatures when the system has settled down to an “equilibrium” state. We have concluded that the temperature of the Hawking radiation for the two-dimensional charged black hole is a function of charge and mass as it is in the case of four-dimensional black hole, in contradistinction to the two-dimensional “Schwarzschild” case where temperature does not depend on mass. We have shown that as the mass of the two-dimensional charged black hole decreases, the temperature of Hawking radiation also decreases till the moment when the mass becomes equal to the charge (extremal case) at which moment the two-dimensional black hole “freezes” completely. This denotes that the extremal black holes can be quantum mechanically a stable ending point for the black holes during the process of their evaporation.

It is worth exploring how the analytical results discussed in this work can be applied for the study of thermodynamic properties of the two-dimensional black holes. It is also interesting to see how the calculations performed for the two-dimensional case can be extended for the higher dimensional geometries, and/or include the other charge a black hole can carry (angular momentum). We hope to return to this issues in a future work.

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